Reanalysis Information for Eigenvalues Derived from a Differential Equation Analysis Formulation

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Introduction

IT is becoming increasingly apparent that, as automated design techniques are applied to more advanced problems, efficient reanalysis techniques and methods for computing derivatives of behavior variables (such as stresses, displacements, and buckling loads) with respect to design variables (sizing parameters) are needed in order to keep computer execution times realistic. A number of recent papers[1–5] have dealt with this subject in the context of simultaneous algebraic analysis equations such as might result from either finite difference or finite element idealizations of a structure. In many situations, the analysis equations can conveniently be formulated and solved as a set of first-order ordinary differential equations. It is the purpose of this Note to demonstrate that perturbation methods can be used in a straightforward manner to obtain reanalysis information when an analysis model is formulated in this way. In particular, a perturbation formula for the buckling loads of a general shell of revolution is derived and some results of its use are given.

Theory

A fairly general set of shell of revolution buckling equations has been formulated by Cohen[6] as

$$r\mathbf{y}' + A\mathbf{y} - B\mathbf{z} - \lambda_e \mathbf{F} = -\mu\alpha\mathbf{y} - \mu\beta\mathbf{z} + \mu\mathbf{F} \tag{1a}$$

$$r\mathbf{z}' - C\mathbf{y} + D\mathbf{z} = -\mu\delta\mathbf{z} \tag{1b}$$

The notation in Eqs.(1) is that introduced by Cohen as Eqs. (25) of Ref. 6 except that overbars on α, β, and δ have been omitted here. The coefficient matrices in Eqs. (1) satisfy the self-adjointness conditions

$$A + D^T = r'I, \quad \alpha + \delta^T = 0, \quad B = B^T, \quad C = C^T, \quad \beta = \beta^T \tag{2}$$

In order to solve Eqs. (1) by direct forward integration, the shell usually must be segmented in the meridional direction. If the shell is subjected to ring loadings or has a number of discrete circumferential rings, segmentation is necessary to accommodate these discontinuities in shell loading and geometry. At points of segmentation, the conditions are, in the notation of Ref. 6,

$$r\Delta\mathbf{y} - K\mathbf{z} = \mu\kappa\mathbf{z}, \quad s = s_i \tag{3}$$

where s_i denotes the positions of meridional segmentation. The matrices K and κ are symmetric, and this symmetry is required for self-adjointness. The conditions at the top and bottom of the shell are assumed to be a special case of Eq. (3). Classical support conditions (hinge, roller, free, etc.) can be considered provided they result in a self-adjoint formulation.

The matrices A, B, C, D, α, β, δ, K, and κ are dependent upon parameters which determine a particular shell design. These parameters, which include shell wall thickness t_w, ring diameter D_R, ring spacing S_R, etc., are called the design variables. The object here is to determine changes in the buckling load μ due to changes in the design variables without performing a complete reanalysis. This is a standard topic in perturbation theory.[7] Due to a perturbation in a design variable, the coefficient matrices $A, \ldots, \kappa$ are rewritten in the form

$$A \to A + \varepsilon A_1, \ldots, \kappa \to \kappa + \varepsilon\kappa_1 \tag{4}$$

where ε is a perturbation parameter. The matrices of Eq. (4) are then substituted in Eqs. (1) and (3) to give, for the pth mode

$$r\bar{\mathbf{y}}_p' + A\bar{\mathbf{y}}_p - B\bar{\mathbf{z}}_p - \lambda_e\bar{\mathbf{F}}_p + \varepsilon A_1\bar{\mathbf{y}}_p - \varepsilon B_1\bar{\mathbf{z}}_p = -\bar{\mu}_p\alpha\bar{\mathbf{y}}_p - \bar{\mu}_p\beta\bar{\mathbf{z}}_p + \bar{\mu}_p\bar{\mathbf{F}}_p - \varepsilon\bar{\mu}_p\alpha_1\bar{\mathbf{y}}_p - \varepsilon\bar{\mu}_p\beta_1\bar{\mathbf{z}}_p \tag{5a}$$

$$r\bar{\mathbf{z}}_p' - C\bar{\mathbf{y}}_p + D\bar{\mathbf{z}}_p - \varepsilon C_1\bar{\mathbf{y}}_p + \varepsilon D_1\bar{\mathbf{z}}_p = -\bar{\mu}_p\delta\bar{\mathbf{z}}_p - \varepsilon\bar{\mu}_p\delta_1\bar{\mathbf{z}}_p \tag{5b}$$

and

$$r\Delta\bar{\mathbf{y}}_p - K\bar{\mathbf{z}}_p - \varepsilon K_1\bar{\mathbf{z}}_p = -\bar{\mu}_p\kappa\bar{\mathbf{z}}_p - \varepsilon\bar{\mu}_p\kappa_1\bar{\mathbf{z}}_p \tag{6}$$

The solution to Eqs. (5) and (6) is assumed to be of the form

$$\begin{aligned} \bar{\mathbf{y}}_p &= \mathbf{y}_p + \varepsilon\boldsymbol{\eta}_p + \cdots, \quad & \bar{\mathbf{z}}_p &= \mathbf{z}_p + \varepsilon\boldsymbol{\zeta}_p + \cdots \\ \bar{\mu}_p &= \mu_p + \varepsilon\nu_p + \cdots, \quad & \bar{\mathbf{F}}_p &= \mathbf{F}_p + \varepsilon\mathbf{Q}_p + \cdots \end{aligned} \tag{7}$$

The solution as given by Eqs. (7) is substituted into Eqs. (5) and (6). Then, grouping terms which have the same power of ε, the result for ε^0, is of the same form as Eqs. (1) and (3) while, for ε^1:

$$r\boldsymbol{\eta}_p' + A\boldsymbol{\eta}_p - B\boldsymbol{\zeta}_p - \lambda_e\mathbf{Q}_p + \mu_p(\alpha\boldsymbol{\eta}_p + \beta\boldsymbol{\zeta}_p - \mathbf{Q}_p) = -A_1\mathbf{y}_p + B_1\mathbf{z}_p - \mu_p(\alpha_1\mathbf{y}_p + \beta_1\mathbf{z}_p) - \nu_p(\alpha\mathbf{y}_p + \beta\mathbf{z}_p - \mathbf{F}_p) \tag{8a}$$

$$r\boldsymbol{\zeta}_p' - C\boldsymbol{\eta}_p + D\boldsymbol{\zeta}_p + \mu_p\delta\boldsymbol{\zeta}_p = C_1\mathbf{y}_p - D_1\mathbf{z}_p - \mu_p\delta_1\mathbf{z}_p - \nu_p\delta\mathbf{z}_p \tag{8b}$$

$$r\Delta\boldsymbol{\eta}_p - K\boldsymbol{\zeta}_p - K_1\mathbf{z}_p = -\mu_p\kappa\boldsymbol{\zeta}_p - \nu_p\kappa\mathbf{z}_p - \mu_p\kappa_1\mathbf{z}_p \tag{8c}$$

and equations for higher powers of ε can be obtained by including higher order terms in Eqs. (7).

In order to derive an expression for ν_p the following steps are taken: 1) premultiply the first of Eqs. (8) by $\mathbf{z}_q^T$, where q represents any other mode, 2) premultiply the second of Eqs. (8) by $-\mathbf{y}_q^T$, 3) add the results of steps 1) and 2) and set equal to zero. Then 4) integrate the expression in step 3) over the shell meridian using the last of Eqs. (8), Eqs. (2), the appropriate end conditions, and the fact that $\int \mathbf{Q}_p^T\mathbf{z}_q\,ds = \int \mathbf{F}_q^T\boldsymbol{\zeta}_p\,ds$, to get, after setting $p = q$,

$$\nu_p = \left\{\int \{2\mathbf{z}_p^T A_1\mathbf{y}_p - \mathbf{z}_p^T B_1\mathbf{z}_p + \mathbf{y}_p^T C_1\mathbf{y}_p\}\,ds + \mu_p\int \{2\mathbf{z}_p^T\alpha_1\mathbf{y}_p + \mathbf{z}_p^T\beta_1\mathbf{z}_p\}\,ds - \sum_{s_i}\mathbf{z}_p^T K_1\mathbf{z}_p - \mu_p\sum_{s_i}\mathbf{z}_p^T\kappa_1\mathbf{z}_p\right\} \Bigg/ \left\{\int \{-2\mathbf{z}_p^T\alpha\mathbf{y}_p - \mathbf{z}_p^T\beta\mathbf{z}_p + \mathbf{F}_p^{T'}\mathbf{z}_p\}\,ds + \sum_{s_i}\mathbf{z}_p^T\kappa\mathbf{z}_p\right\} \tag{9}$$

The perturbations in $\mathbf{y}$ and $\mathbf{z}$ ($\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$) can be obtained using as a guide the steps leading to Eq. (9) and the approach of Ref. 7. Then, higher order terms in the expansions for $\mathbf{y}$, $\mathbf{z}$, and μ can be obtained. Also, note that Eq. (9) is not valid for repeated eigenvalues. The modifications which must be made for repeated eigenvalues are discussed in Ref. 7.

Results

To test the accuracy of Eq. (9) and to gain some insight into its range of usefulness, a particular stiffened shell of revolution was studied. The shell considered was a 70° cone with the geometry and boundary conditions shown in Fig. 1 and the material and stiffener configuration of Example 7 of Ref. 8.

The shell was modeled by Eqs. (1) and solved by forward integration using 6 segments (Fig. 1) for accuracy. In each segment the design variables were allowed to vary independently of the design variables in any other segment, except for stringer spacing S_S, which for continuous stringers must conform to the shell geometry. This shell was optimized for weight subject to local stress and buckling constraints similar to those of Ref. 8 and the resulting design is given in Table 1. This shell weighs 48.4 lb, but the general buckling load, denoted λ_{cr_0}, is 2.06 psi with circumferential mode number 6. Since the actual load is

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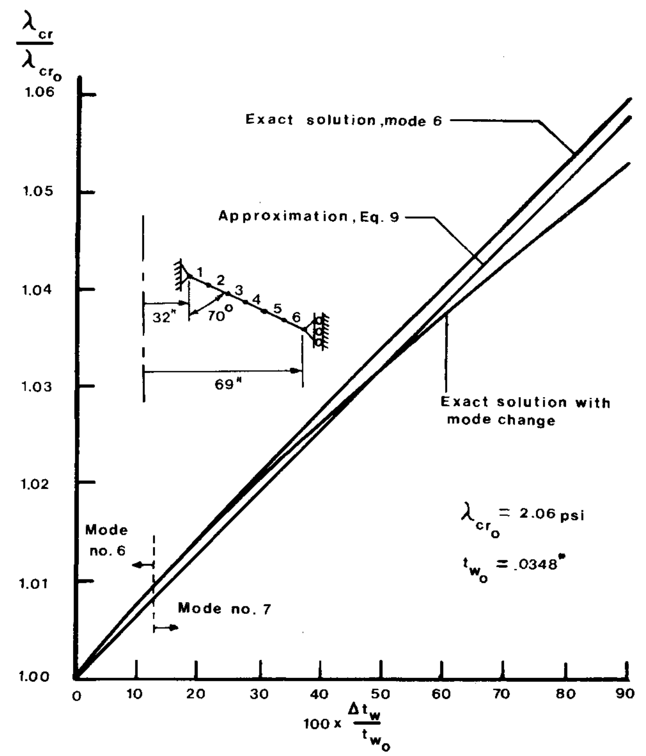

Fig. 1 Shell geometry and effect on buckling load of change in thickness of shell, segment 3.

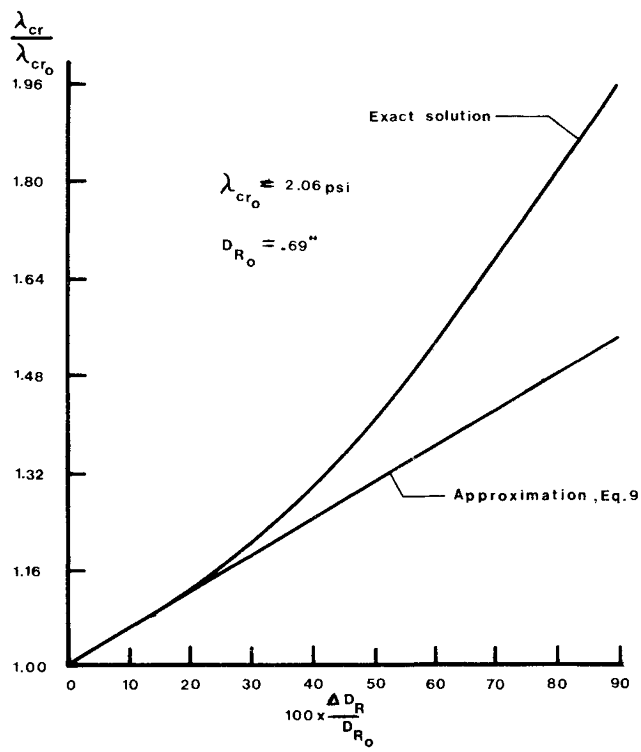

Fig. 2 Effect on buckling load of change in ring diameter, segment 3.

3.75 psi, the design is unsatisfactory. Information on how to increase the general buckling load with the least increase in weight can be obtained through the use of Eq. (9). Considering first the wall thickness t_w, if the matrices A_1, B_1, etc. are taken as $\partial A/\partial t_w$, $\partial B/\partial t_w$, etc., then $v_6 = \partial \lambda_{cr_o}/\partial t_w$. Considering each design variable and each segment in turn, a matrix of derivatives $\partial \lambda_{cr_o}/\partial D_{ij}$, where D_{ij} is the ith design variable in the jth segment, can be computed, as shown in Table 2 (derivatives for t_S and t_R are not included in Table 2 because these quantities are computed by local buckling or minimum gage considerations as noted in Table 1. The magnitude of an element in Table 2 indicates how effective changing the corresponding design variable will be in changing the buckling load. However, since t_w is an order of magnitude smaller than any other design variable in Table 2, its derivative would have to be correspondingly larger than the other derivatives to be as effective. With this in mind it is clear that increasing the ring diameter will give the greatest payoff in increasing the buckling load, and that the rings in the 2nd, 3rd, and 4th segments will have a greater effect than the rings in the other segments.

Table 1 Design variables for piecewise uniform 70° cone design

Shell segment	Wall thickness t_w in.	Stringer thickness t_S in.[a]	Ring thickness t_R in.[b]	Stringer depth D_S in.	Ring diameter D_R in.	Stringer spacing S_S in.[d]	Ring spacing S_R in.
1	0.0400	0.0232	0.01[c]	0.8112	0.6620	3.1569	1.932
2	0.0431	0.0231	0.01[c]	0.8063	0.667	3.1569	1.822
3	0.0348	0.0231	0.01[c]	0.8052	0.6900	3.1569	1.722
4	0.0638	0.0265	0.01[c]	0.9233	0.6242	3.1569	1.676
5	0.0306	0.0325	0.01[c]	1.134	0.6315	3.1569	1.692
6	0.0220	0.0334	0.01[c]	1.164	0.5934	3.1569	2.028

[a] $t_S = \max\{(\sigma_{yp}/3.35E)^{1/2}D_S, 0.01\}$
[b] $t_R = \max\{(\sigma_{yp}/0.4E)D_R, 0.01\}$
[c] Minimum gage.
[d] Stringer spacing at small end of cone.

Table 2 Derivatives of λ_{cr_o} with respect to design variables

Shell segment	Wall thickness t_w	Stringer depth D_S	Ring diameter D_R	Stringer spacing S_S	Ring spacing S_R
1	0.7186	0.0009	0.1018	−0.0179	−0.0097
2	0.7327	0.0232	0.3111	−0.0179	−0.0306
3	1.070	0.1491	0.4794	−0.0179	−0.0497
4	0.8298	0.1785	0.3530	−0.0179	−0.0352
5	0.8556	0.0527	0.1848	−0.0179	−0.0187
6	0.8276	0.0092	0.0486	−0.0179	−0.0040

The wall thickness t_w and ring diameter D_R of segment 3 were chosen to study the accuracy of Eq. (9) when large changes in these variables are considered. Figure 1 shows the comparison between the linear approximation given by Eq. (9) as $\lambda_{cr_{approx}} = \lambda_{cr_o} + (\partial \lambda_{cr_o}/\partial t_{w_3})\Delta t_{w_3}$ with the exact solution. Note that at about 15% change in t_{w_3} the mode shape of the critical load changes from 6 to 7. Considering mode number change an error of 0.5% in the buckling load calculation resulted from a 90% change in t_{w_3}. Figure 2 shows a comparison of the exact and linear approximation values for λ_{cr} when D_{R_3} is changed. No mode number change occurred here. A 90% increase in D_{R_3} resulted in a 42% error in the buckling load using the linear approximation. This is a fairly large error and techniques for reducing it, similar to those presented in Ref. 4, should be developed for this type of formulation.

Observing that $3.75/2.06 = \lambda_{cr}/\lambda_{cr_o} = 1.82$, Fig. 2 indicates that an 80% increase in D_{R_3} will yield a design with a general buckling load of 3.75 psi. This design was found to be acceptable with respect to local yielding and buckling constraints and weighs 49.7 lb.

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Least Squares Nonlinear Parameter Estimation by the Iterative Continuation Method

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Introduction

THE principle of least squares is widely used for estimating the unknown parameters of a nonlinear analytic model from a discrete sequence of noisy observations. In applying this principle, estimates of the parameters are determined by minimizing an error function based on the squares of the deviations between the computed response and the measured response. Minimization of the error function may be performed by several iterative procedures such as the linearization method.[1,2] This method has the advantage of a fast convergence, however, it necessitates good starting values for converging. It is proposed here to use a continuation method for finding good first approximations for the linearization procedure. Continuation methods are of two kinds: continuous or iterative.[3] The proposed method is of the second type. Its convergence properties will be illustrated by a numerical example.

A continuation method has been recently used for fitting a differential equation to experimental data.[4] The unknown parameters were determined by solving a nonlinear multipoint boundary-value problem. The solution is based on a continuation method of the continuous type.

Analysis

The system to identify is described by the equation

$$y = f(x, \gamma) \tag{1}$$

where f is a nonlinear function of the unknown parameter vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ and of the independent variable x.

Noisy measurements b_m are made on the variable $y(x)$ at stations x_m, $m = 1, 2, \ldots, N$. The estimation problem is to find the parameter vector $\hat{\gamma}$ which minimizes the function

$$J(\gamma) = \sum_{m=1}^{N} [b_m - f(x_m, \gamma)]^2 \tag{2}$$

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The concept of continuation will be applied to solve this minimum problem. As said in Ref. 5, "continuation refers to the situation in which a problem depends on a parameter and in which the solution to the problem for one value of the parameter is used to find the solution for a nearby value of the parameter." Sometimes, this parameter t appears in the definition of the problem and the application of the continuation method is straightforward. However, if the parameter t does not appear explicitly in the problem, it is necessary in applying the continuation method to introduce it.

An auxiliary parameter t will be introduced in the problem in the following manner. Let γ^0 be an initial approximation of the minimizing vector $\hat{\gamma}$ of the function $J(\gamma)$. Define the function

$$S(\gamma, t) = tJ(\gamma) + (1-t)\sum_{m=1}^{N} [c_m(t) - f(x_m, \gamma)]^2 \tag{3}$$

where $J(\gamma)$ is defined in Eq. (2) and

$$c_m(t) = tb_m + (1-t)f(x_m, \gamma^0) \tag{4}$$

The functions $c_m(t)$ appear in Eq. (11) of Wasserstrom's paper.[4] They are similarly used in the present formulation.

The problem $P(t)$ is now considered. The problem is to find the minimizing vector $\hat{\gamma}(t)$ of the function $S(\gamma, t)$ for t in the interval $[0, 1]$. At $t = 0$, this problem is easily solved. The function $S(\gamma, 0)$ is

$$S(\gamma, 0) = \sum_{m=1}^{N} [f(x_m, \gamma^0) - f(x_m, \gamma)]^2 \tag{5}$$

The minimizing vector of $S(\gamma, 0)$ is trivial,

$$\hat{\gamma}(0) = \gamma^0 \tag{6}$$

At $t = 1$, the following condition applies:

$$S(\gamma, 1) = J(\gamma) \tag{7}$$

Therefore, $P(1)$ coincides with the original minimum problem and

$$\hat{\gamma}(1) = \hat{\gamma} \tag{8}$$

The desired solution $\hat{\gamma}(1)$ is next computed by a continuation procedure which starting from γ^0 at $t = 0$ "continues" this initial solution and yields $\hat{\gamma}$ at $t = 1$. Using the discrete version of the continuation method, the interval $0 \leq t \leq 1$ is subdivided into M equal parts so that $t_i = i/M$, $i = 1, 2, \ldots, M$. Each problem $P(t_i)$ will be solved by the linearization method. In applying this latter method, the solution $\hat{\gamma}(t_{i-1})$ to the problem $P(t_{i-1})$ is chosen as a starting value for solving the next problem $P(t_i)$. In particular, $\hat{\gamma}(0) = \gamma^0$ is taken as a first approximation for solving the problem $P(t_1)$.

The solution of the problem $P(t_i)$ by the linearization method is next described. For reducing the notation define

$$\hat{\gamma}(t_i) = \gamma^i \qquad i = 1, 2, \ldots, M$$

$$f_l(x, \gamma) = \partial f(x, \gamma)/\partial \gamma_l \qquad l = 1, 2, \ldots, k$$

Assume that γ^{i-1} is close to γ^i, then

$$\gamma^i \sim \gamma^{i-1} + \Delta\gamma \tag{9}$$

where $\Delta\gamma = (\Delta\gamma_1, \Delta\gamma_2, \ldots, \Delta\gamma_k)$ is a vector of small corrections. Linearize the function $f(x, \gamma)$ about γ^{i-1}

$$f(x, \gamma) \sim f(x, \gamma^{i-1}) + \sum_{l=1}^{k} f_l(x, \gamma^{i-1})\Delta\gamma_l \tag{10}$$

Substituting Eq. (10) into Eq. (3) yields the function

$$SS(\Delta\gamma, t_i) = t_i \sum_{m=1}^{N} [b_m - f(x_m, \gamma^{i-1}) - \sum_{l=1}^{k} f_l(x_m, \gamma^{i-1})\Delta\gamma_l]^2 +$$
$$(1-t_i)\sum_{m=1}^{N} [c_m(t_i) - f(x_m, \gamma^{i-1}) - \sum_{l=1}^{k} f_l(x_m, \gamma^{i-1})\Delta\gamma_l]^2 \tag{11}$$

$SS(\Delta\gamma, t_i)$ is a quadratic form in $\Delta\gamma$ whose minimizer is

$$\Delta\gamma = A^{-1}B \tag{12}$$

The elements of $A(k \times k)$ and $B(k \times 1)$ are given by

$$A_{jh} = \sum_{m=1}^{N} f_h(x_m, \gamma^{i-1}) f_j(x_m, \gamma^{i-1}) \tag{13}$$